

# Dynamics of a planar Coulomb gas

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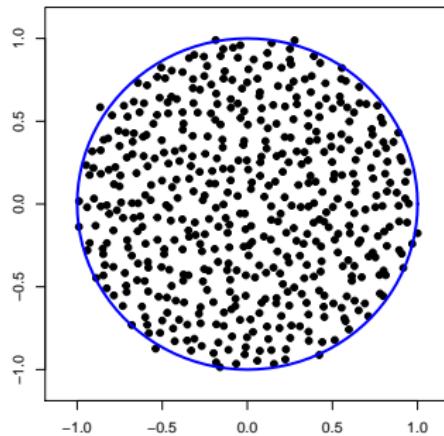
Workshop on Optimal and Random Point Configurations  
February 26, 2018 – ICERM, Brown University

## Joint work with...



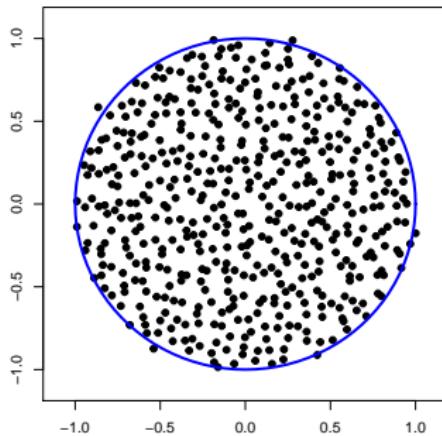
François BOLLEY and Joaquín FONTBONA

## Motivation: Ginibre Ensemble



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Stochastic process leaving invariant this random picture?
```

## Outline

Poincaré inequality

Dyson Process

Ginibre process

## Markov diffusion processes

- $H : \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $H(x)$  energy of state or configuration  $x \in \mathbb{R}^d$

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- Gradient dynamical system with noise

$$x_{n+1} - x_n = -\nabla H(x_n) + g_n$$

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- Decay to the equilibrium: for all  $p_0$ ,

$$\partial_t \|p_t - 1\|_{L^2(\mu)}^2 = \partial_t \text{Var}_\mu(p_t) = 2\mathbb{E}_\mu(p_t G p_t) = -2\mathbb{E}_\mu(|\nabla p_t|^2) \leq 0.$$

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- Exponential decay equivalent to Poincaré inequality: for all  $\rho > 0$ ,

$$\forall p_0, t, \text{Var}_\mu(p_t) \leq e^{-2\rho t} \text{Var}_\mu(p_0) \quad \text{iif} \quad \forall f, \text{Var}_\mu(f) \leq -\frac{\mathbb{E}_\mu(|\nabla f|^2)}{\rho}.$$

## Exactly solvable model: Ornstein–Uhlenbeck process

■ Gaussian model:  $H(x) = \frac{|x|^2}{2}$ ,  $dX_t = -X_t dt + dB_t$ ,

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$$GP_n = -nP_n \quad \text{and} \quad G(\cdot) = -\sum_{n=0}^{\infty} n \langle \cdot, P_n \rangle P_n$$

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- Optimal Poincaré inequality, equality achieved for  $f = P_1$

$$\text{Var}_{\mu}(f) \leq \mathbb{E}_{\mu}(|\nabla f|^2).$$

## Comparison to Gaussianity via convexity

Theorem (Brascamp–Lieb 1976)

If  $\mu(dx) = \frac{e^{-H(x)}}{Z} dx$ ,  $\nabla^2 H > 0$  on  $\mathbb{R}^d$ , then for any smooth  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ ,

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- Jensen divergence:  $\text{Var}_\mu(f) = \mathbb{E}_\mu \Phi(f) - \Phi(\mathbb{E}_\mu f)$ ,  $\Phi(u) = u^2$

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- Logarithmic Sobolev:  $I = \mathbb{R}_+$ ,  $\Phi(u) = u \log(u)$

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### Theorem (Caffarelli 2000)

If  $\mu(dx) = \frac{e^{-H(x)}}{Z} dx$ ,  $\nabla^2 H \geq \rho I_d > 0$ , then  $\mu$  is the image of  $\mathcal{N}(0, I_d)$  by a Lipschitz function  $F : \mathbb{R}^d \rightarrow \mathbb{R}^d$  with  $\|F\|_{\text{Lip}} \leq \frac{1}{\sqrt{\rho}}$ .

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- Gives also any  $\Phi$ -Sobolev inequality from the Gaussian!

## KLS conjecture

Conjecture (Kannan–Lovász–Simonovits 1995)

*There exists a universal constant  $C > 0$  such that for any dimension  $d \geq 1$  and any smooth  $H : \mathbb{R}^d \rightarrow \mathbb{R}$  with  $\nabla^2 H \geq 0$  and  $\text{Cov} = I_d$ ,  $\mu(dx) = \frac{e^{-H(x)}}{Z} dx$  satisfies to a Poincaré inequality with constant  $C$ .*



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- Stochastic process of spectrum?

# Gaussian Unitary Ensemble and Dyson Process

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## Outline

Poincaré inequality

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$$z \in \mathbb{C} \mapsto \varphi(z) = \frac{e^{-n|z|^2}}{\pi} \sum_{\ell=0}^{n-1} \frac{n^\ell |z|^{2\ell}}{\ell!}.$$

- Circular law

$$\lim_{n \rightarrow \infty} \sup_{z \in K} \left| \varphi(z) - \frac{\mathbf{1}_{\{|z| \leq 1\}}}{\pi} \right| = 0.$$

- The function  $z \mapsto \log \sum_{\ell=0}^{n-1} \frac{|z|^{2\ell}}{\ell!}$  is concave!
- Second moment of  $\varphi$  bounded in  $n$  then KLS/Bobkov theorem

## Second moment dynamics

Theorem (Second moment dynamics)

$(R_t)_{t \geq 0} = \left( \frac{|X_t|^2}{n} \right)_{t \geq 0}$  is an ergodic Cox–Ingersoll–Ross process:

$$dR_t = 4 \frac{\alpha_n}{n} \left[ \frac{n}{\beta_n} + \frac{n-1}{2n} - R_t \right] dt + \sqrt{\frac{4\alpha_n}{n\beta_n}} R_t dB_t.$$

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In particular, with  $\Gamma_n = \text{Gamma}(n + \frac{n-1}{2n}\beta_n, \beta_n)$ , for any  $t \geq 0$

$$W_1(\text{Law}(R_t), \Gamma_n) \leq e^{-4\frac{\alpha_n}{n}t} W_1(\text{Law}(R_0), \Gamma_n).$$

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Moreover for any  $x \in D$  and  $t \geq 0$ , we have

$$\mathbb{E}(R_t | R_0 = r) = r e^{-\frac{4\alpha_n}{n} t} + \left( \frac{1}{2} + \frac{n}{\beta_n} - \frac{1}{2n} \right) \left( 1 - e^{-\frac{4\alpha_n}{n} t} \right).$$

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If  $\sigma = \lim_{n \rightarrow \infty} \frac{\alpha_n}{\beta_n} \in [0, \infty)$  then  $\lim_{n \rightarrow \infty} v_{n,t} = v_t$  with . . .

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- Without noise and confinement: . . . , Duerinckx (2016), . . .

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- → Hamiltonian or Hybrid Monte Carlo (HMC, C.-Ferré-Stoltz)



That's all folks!

Thank you for your attention.